On the resummation of the $\alpha \ln^2 z$ terms for QED corrections to deep-inelastic ep scattering and e^+e^- annihilation

J. Blümlein¹, S. Riemersma¹, A. Vogt²

¹ DESY-Zeuthen, Platanenallee 6, D-15735 Zeuthen, Germany

² Institut für Theoretische Physik, Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany

Received: 5 November 1996 / Revised version: 16 January 1997

Abstract. The resummation of the $\alpha \ln^2(z)$ non-singlet contributions is performed for initial state QED corrections. As examples, the effect of the resummation on neutral-current deep-inelastic scattering and the $e^+e^- \to \mu^+\mu^-$ scattering cross section near the Z^0 -peak is investigated.

1 Introduction

The non-singlet splitting functions of QCD are known to behave as $\alpha_s^{l+1} \ln^{2l}(z)$ [1] for small values of z, the momentum fraction determining the corresponding radiator function. A similar behaviour is observed also in QED¹ [3, 4]. These terms may potentially yield large contributions to the radiative corrections. In an approach based on the systematic evaluation of the Feynman diagrams at a fixed order in the coupling constant, the contributions of $O[\alpha^{l+1}]$ $\ln^{2l}(z)$ emerge from a wide class of terms, see for example [3,5]. Therefore the all-order resummation of these terms cannot be carried out by direct diagram calculations but is performed by solving so-called infrared evolution equations [1]. It is interesting to note that double logarithmic terms emerge also in other physical situations. As an example we mention the Sudakov form factor, which is mostly studied for the region $z \sim 1$. For a fixed coupling it behaves $\propto \alpha \ln^2(Q^2/m^2)$, see [6]. Contributions to this quantity $\propto \ln^2 z$ may become important as $z \to 0$.

In the present paper we calculate the contribution of the small-z resummed terms to the initial state radiative corrections for deep-inelastic ep scattering (DIS). We compare these corrections with those resummed by the nonsinglet Altarelli–Parisi equation in QED, $\propto \alpha^l \ln^l(Q^2/m_e^2)$. We also evaluate the contribution of these terms to the initial state corrections to $e^+e^- \to \mu^+\mu^-$ at the Z^0 -peak.

2 Basic relations

The evolution of the non-singlet electron structure function $D(z,Q^2)$ is governed by

$$\frac{\partial D(z,Q^2)}{\partial \ln Q^2} = P\left[z,\alpha(Q^2)\right] \otimes D(z,Q^2), \tag{1}$$

where \otimes denotes the Mellin convolution

$$A(z) \otimes B(z) \equiv \int_0^1 \int_0^1 dz_1 dz_2 A(z_1) B(z_2) \delta(z - z_1 z_2).$$
 (2)

Equation (1) is a flavor non-singlet evolution equation, which results from the renormalization group equation [7] ruling the mass factorization [8]. As well-known this equation can be equivalently derived using the operator product expansion [9], which is valid for $z \in [0,1]$, if $P(z,\alpha)$ and $D(z,Q^2)$ are interpreted as distributions [10]. This we will do in the following.

The splitting function $P\left[z,\alpha(Q^2)\right]$ can be represented by the series

$$P\left[z,\alpha(Q^2)\right] = \sum_{k=1}^{\infty} a^k(Q^2) P_k(z), \tag{3}$$

with $a(Q^2) = \alpha(Q^2)/(4\pi)$. In leading order, the evolution of the QED coupling constant $a(Q^2)$ is described by

$$\frac{\partial a(Q^2)}{\partial \ln Q^2} = \frac{4}{3}a^2(Q^2),\tag{4}$$

yielding

$$a(Q^2) = \frac{a(m_e^2)}{1 - \frac{4}{3}a(m_e^2)\ln\left(\frac{Q^2}{m_e^2}\right)}.$$
 (5)

Here we have considered only the electron threshold in the evolution. For the solution of (1), we use the first-order splitting function²

$$P_1(z) = 2\left(\frac{1+z^2}{1-z}\right)_{\perp}$$
 (6)

 $^{^1\,}$ A first application to QED was discussed in [2], considering forward $e^+e^-\to \mu^+\mu^-$ annihilation in the high energy limit

 $^{^2}$ This solution of the evolution equation is well-known from QCD [11]. For the non-singlet case the $O(\alpha \ln(Q^2/m^2))$ result for the initial state QED corrections to e^+e^- annihilation was obtained in [12] already. Later applications were given, e.g., in [13]

For the higher order contributions in $a(Q^2)$, we account for the leading terms at small z, denoted by $P_{z\to 0}$, which are mation $P_{z\to 0}(z,a) = \sum_{k=0}^{\infty} c_k a^{k+1} \ln^{2k}(z)$ $\propto a^{l+1} \ln^{2l}(z)$. The latter terms are obtained in resummed form in Mellin space by

$$\mathcal{M}[P_{z\to 0}](N,a) \equiv \int_0^1 dz \ z^{N-1} P_{z\to 0}(z,a)$$
$$\equiv -\frac{1}{2} \Gamma_{z\to 0}^-(N,a) = \frac{1}{8\pi^2} f_0^-(N,a).$$
(7)

 $f_0^-(N,a)$ is the solution to the equation [1]

$$f_0^-(N,a) = 16\pi^2 \frac{a}{N} + 8\frac{a}{N^2} f_V^+(N,a) + \frac{1}{8\pi^2} \frac{1}{N} \left[f_0^-(N,a) \right]^2, \tag{8}$$

and $f_V^+(N,a)$ obeys

$$f_V^+(N,a) = 16\pi^2 \frac{a}{N} + \frac{1}{8\pi^2} \frac{1}{N} \left[f_V^+(N,a) \right]^2 .$$
 (9)

Here the coefficients in (8, 9), originally given for SU(N)in [1], were adjusted to the case of QED, see [4].

For the resummed anomalous dimension, one finally obtains

$$\begin{split} \pmb{\varGamma}_{z \to 0}^{-, \text{QED}}(N, a) \\ &= -N \left\{ 1 - \sqrt{1 + \frac{8a}{N^2} \left[1 - 2\sqrt{1 - \frac{8a}{N^2}} \; \right]} \right\} \; . \; (10) \end{split}$$

 $\mathcal{M}[P_{z\to 0}](N,a)$ can be represented in terms of a Taylor series in a by

$$\mathcal{M}[P_{z\to 0}](N,a) = \sum_{k=0}^{\infty} a^{k+1} \frac{p_k}{N^{2k+1}}$$

$$= \frac{2a}{N} - \frac{12a^2}{N^3} - \frac{80a^3}{N^5} - \frac{304a^4}{N^7} + \dots$$
(11)

The resummed small-z part of the splitting function P(z,a)is obtained transforming (11) back to z-space,

$$P_{z\to 0}(z,a) = \sum_{k=0}^{\infty} c_k a^{k+1} \ln^{2k}(z), \quad c_k = \frac{p_k}{(2k)!}$$
. (12)

The numerical values of the first coefficients c_k are listed in Table 1. Evidently the first term in (11) agrees with the limit $z \to 0$ of the lowest order (LO) splitting function (6). In [4], (23, 26), we showed that the NLO contribution to (11) agrees both with the limit of the splitting function derived in [5] in the $\overline{\rm MS}$ scheme³ and with the result of [3] in the on-mass-shell scheme.

$$D(z, Q_0^2 = m_e^2) = \delta(1 - z) \tag{13}$$

k	c_k
0	2.0000E + 0
1	-6.0000E + 0
2	-3.3333E + 0
3	-0.4222E + 0
4	1.5873E - 3
5	2.8571E - 3
6	1.4000E - 4
7	-3.8468E - 7
8	-2.0649E - 7
9	-6.1484E - 9

as the initial condition for the solution of (1). For the splitting functions $P_k(z)$ in (3),

$$\int_0^1 dz P_k(z) = 0 \tag{14}$$

holds due to fermion number conservation. For the resummed kernel $P_{z\to 0}(z,a)$, (12), the integral condition (14) is not obeyed a priori but has to be restored. In the subsequent treatment we will subtract the term $p_{k-1}\delta(1$ z) in $O(a^k)$.

As outlined in [4,14] for the resummation of the smallx terms for different processes in QCD, less singular terms can be as important as the leading singular terms. In QED, the $O[\alpha^2 \ln(z) \ln(Q^2/m_e^2)]$ terms are known for e^+e^- annihilation [3]. From the different contributions, all terms but the well-known term due to the vacuum polarization function cancel. In $O(\alpha^2)$ the respective correction is

$$-12\frac{a^2}{N^3}\left(1-\frac{2}{9}N\right).$$

In this order, the coefficient of the term being one order less singular in 1/N is much smaller than that of the leading term.

3 Non-singlet QED radiative corrections to deeply inelastic ep scattering

The Born cross section for neutral-current deep-inelastic ep scattering is given by

$$\frac{d^2 \sigma_{NC}^B}{dx \, dy} = \frac{2\pi \alpha^2 S}{Q^4} \left[Y_+ \mathcal{F}_2(x,Q^2) + Y_- x \mathcal{F}_3(x,Q^2) \right], (15)$$

with $Y_{\pm} = 1 \pm (1 \pm y)^2$, x and y are the Bjorken variables, S is the cm energy squared, $Q^2 = xyS$ and

$$\mathcal{F}_2(x, Q^2) = F_2(x, Q^2) + 2|Q_e|(v_e + \lambda a_e)\chi(Q^2)G_2(x, Q^2)$$

³ The QED result is obtained setting the color factors in [5] to $C_F = T_F = 1$ and $C_G = 0$, see [4]

$$+4(v_e^2 + a_e^2 + 2\lambda v_e a_e)\chi^2(Q^2)H_2(x, Q^2)$$
 (16)

$$x\mathcal{F}_{3}(x,Q^{2})$$

$$= -2 \operatorname{sign}(Q_{e}) \Big\{ |Q_{e}|(a_{e} + \lambda v_{e})\chi(Q^{2})xG_{3}(x,Q^{2}) - (17 + [2v_{e}a_{e} + \lambda(v_{e}^{2} + a_{e}^{2})]\chi^{2}(Q^{2})xH_{3}(x,Q^{2}) \Big\},$$

with $Q_e=-1$ for electron and $Q_e=1$ for positron scattering. $\lambda=\xi \, \mathrm{sign}(Q_e)$ denotes the lepton polarization, $v_e=1-4\sin^2\theta_W, a_e=1,\,\theta_W$ the weak mixing angle, and

$$\chi(Q^2) = \frac{G_F}{\sqrt{2}} \frac{M_Z^2}{8\pi\alpha^2} \frac{Q^2}{Q^2 + M_Z^2}.$$
 (18)

 G_F is the Fermi constant and M_Z the mass of the Z^0 -boson. The neutral-current structure functions in (16, 17) are described in the parton model by

$$F_2(x,Q^2) = x \sum_{i=1}^{N_f} e_i^2 [q_i(x,Q^2) + \overline{q}_i(x,Q^2)], \qquad (19)$$

$$G_2(x, Q^2) = x \sum_{i=1}^{N_f} |e_i| v_i [q_i(x, Q^2) + \overline{q}_i(x, Q^2)],$$
 (20)

$$H_2(x, Q^2) = \frac{x}{4} \sum_{i=1}^{N_f} (v_i^2 + a_i^2) [q_i(x, Q^2) + \overline{q}_i(x, Q^2)], \tag{21}$$

$$xG_3(x,Q^2) = x \sum_{i=1}^{N_f} |e_i| a_i [q_i(x,Q^2) - \overline{q}_i(x,Q^2)], \quad (22)$$

$$xH_3(x,Q^2) = \frac{x}{2} \sum_{i=1}^{N_f} v_i a_i [q_i(x,Q^2) - \overline{q}_i(x,Q^2)], \quad (23)$$

with $v_i = 1 - 4|e_i|\sin^2\theta_W$, $a_i = 1$, N_f the number of flavors, and q_i , \bar{q}_i denote the quark and antiquark densities, respectively.

The QED radiative corrections due to initial state electron radiation can be expressed by

$$\frac{d^2 \sigma_{NC}^{isr}}{dx \, dy} = \frac{d^2 \sigma_{NC}^B}{dx \, dy} + \int_0^1 dz D(z, Q^2)$$

$$\times \left\{ \theta(z - z_0) \mathcal{J}(x, y, z) \frac{d^2 \sigma_{NC}^B}{dx \, dy} \Big|_{x = \hat{x}, y = \hat{y}, s = \hat{s}} - \frac{d^2 \sigma_{NC}^B}{dx \, dy} \right\},$$
(24)

with $D(z, Q^2)$ the solution of (1) for z < 1, and

$$\mathcal{J}(z,y,z) = \begin{vmatrix} \partial \hat{x}/\partial x \ \partial \hat{y}/\partial x \\ \partial \hat{x}/\partial y \ \partial \hat{y}/\partial y \end{vmatrix} . \tag{25}$$

 $D(z,Q^2)$ receives contributions from the iteration of the non-singlet kernel $R_1(z) = P_1(z)|_{z<1}$, which are obtained by

$$D_{AP}(z,Q^2) = \sum_{k=1}^{\infty} \frac{1}{k!} \zeta^k(Q^2) \otimes_{l=1}^k R_1(z), \qquad (26)$$

where

$$\zeta(Q^2) = -\frac{3}{2} \ln \left[1 - \frac{4}{3} a_0 \ln \left(\frac{Q^2}{m_e^2} \right) \right]. \tag{27}$$

In the subsequent numerical calculation, we evaluate the initial state radiative corrections for the case of leptonic variables [15,16]. Here the shifted quantities $\hat{x}, \hat{y}, \hat{s}$ and the threshold z_0 are

$$\hat{x} = \frac{xyz}{z+y-1}, \quad \hat{y} = \frac{z+y-1}{z},$$

$$\hat{s} = zS, \quad z_0 = \frac{1-y}{1-xy}.$$
(28)

The contributions to (26) are taken into account up to k = 3 completely⁴. For the higher-order terms we add the solution of (1) in the soft limit

$$D_{AP}^{\text{soft}}(z, Q^2)|_{(4)}$$

$$= 2\zeta (1-z)^{2\zeta-1} \frac{\exp[\zeta(\frac{3}{2}-2\gamma_E)]}{\Gamma(1+2\zeta)} - \frac{2\zeta}{1-z}$$

$$- [3+4\ln(1-z)] \frac{\zeta^2}{1-z}$$

$$- \left[4\ln^2(1-z) + 6\ln(1-z) + \frac{9}{4} - \frac{2\pi^2}{3}\right] \frac{\zeta^3}{1-z}$$

for z < 1. The regularization is inherent in (24). The contribution of the small-z resummed terms to $D(z, Q^2)$ is

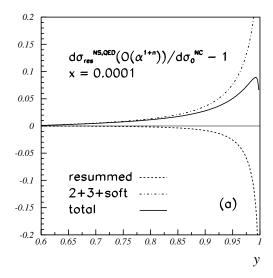
$$D_{z\to 0}(z,Q^2) = \sum_{k=1}^{\infty} c_k \int_{m_e^2}^{Q^2} \frac{dq^2}{q^2} a^{k+1}(q^2) \ln^{2k}(z). \quad (30)$$

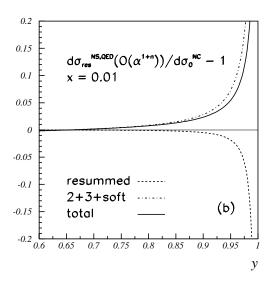
In Fig. 1a–c, we show the contributions to the initial state QED corrections to $d^2\sigma_{NC}^B/dxdy$ in the kinematic range of HERA starting with the terms in $O(\alpha^2)$ to allow for a better comparison. The first order corrections are well-known, see [18,15]. We compare the small-z resummed terms to those obtained by iterating the kernel $R_1(z)$. The small-z resummed terms are negative and contribute only for large values of y. There they diminish the positive leading order corrections $O[\alpha^l \ln^l(Q^2/m_e^2)]$ significantly. These corrections are therefore relevant and need to be considered in the case of high y measurements, such as the determination of the structure function $F_L(x,Q^2)$ in the small-x range. For larger values of x, the small-z resummed corrections contribute only for the highest values of y.

4 $\,lpha \ln^2(z)$ QED corrections to the Z^0 peak

A second important application of the small-z resummation concerns its possible effect upon the $e^+e^- \to \mu^+\mu^-$ cross section near the Z^0 -peak. The implications would be quite profound were the resummation-improved cross section to have a measurable impact on the total cross

⁴ Analytic results for the convolutions of $R_1(z)$ are easily obtained, see [16] and [17] for explicit expressions





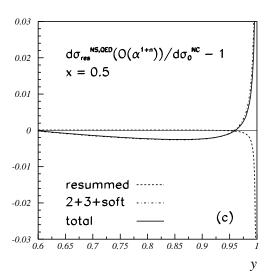


Fig. 1. Second and higher order initial state radiative corrections to deep inelastic ep scattering at HERA, $\sqrt{s}=314$ GeV. Dashed lines: contribution due to the resummed small–z terms; dash–dotted lines: contribution due to the solution of the Altarelli–Parisi equation with the leading order NS–evolution kernel accounting for the complete $O(\alpha^2L^2)$ and $O(\alpha^3L^3)$ terms and the soft-photon exponentiation beyond $O(\alpha^3)$; full lines: resulting correction. a: x=0.0001, b: x=0.01, and c: x=0.5

section or upon the position of the Z^0 -peak or width on the order of an MeV.

The QED corrections up to $O(\alpha^2)$ were calculated in [3]. We consider the initial state corrections which are calculated accounting for the contributions to $O(\alpha^2)$ and soft-photon exponentiated terms using the code ZFITTER [19]. The small-z resummed terms (12) are taken into account for the contributions higher than second order by

$$\sigma_{z\to 0} = 2 \int_0^1 dz \left[\Theta(z - z_0) \sigma_B(zs) - \sigma_B(s) \right] R_{z\to 0}^-(z, s) ,$$
(31)

where $z_0 = s'/s$, s' being the cm energy entering the annihilation, and $\sigma_B(s)$ the Born cross section. The radiator $R_{z\to 0}^-(z,s)$ is given by

$$R_{z\to 0}^{-}(z,s) = \int_{m_e^2}^{s} \frac{ds'}{s'} \sum_{k=3}^{\infty} c_k a^{k+1}(s') \ln^{2k}(z) . \qquad (32)$$

The factor of 2 enters in (31) because of the initial state radiation from both the electron and the positron line.

A series of cuts on s' has been made and the results are listed in Table 2. The parameters of the calculation are $M_Z=91.1887~{\rm GeV},~\Gamma_Z=2.4974~{\rm GeV},~{\rm and}~{\sin}^2\theta_W=0.2319$. The small-z contribution is six orders of magnitude down from the cross section containing the standard QED corrections. A measurement of this effect is clearly out of the question.

We also have compared the maximum of the cross section of ZFITTER as a function of the cm energy with and without the small-z resummation. We find the difference to be smaller than 40 eV, widely independent of the z-cut. Here the effects of the small-z contributions beyond second order are much smaller than the experimental resolution.

5 Conclusions

The resummation of the $O[\alpha \ln^2(z)]$ non-singlet contributions was performed for initial state QED corrections. As examples, we investigated the effects of the resummation

Table 2. Dependence of the cross section $\sigma(e^+e^- \to \mu^+\mu^-)$ at the Z^0 -peak on the minimum energy of the final state muons. σ_R : scattering cross section including the initial state QED corrections to $O(\alpha^2)$ and soft-photon exponentiation; $\sigma_{z\to 0}$: small-z resummed contributions beyond $O(\alpha^2)$, (31)

$E_{min}^{\mu} (\text{GeV})$	5	10	20	40
$\sigma_R \text{ (nb)}$	1.4723	1.4713	1.4702	1.4674
$\sigma_{z\to 0}$ (nb)	$1.05341 \ 10^{-6}$	$1.13476 \ 10^{-6}$	$1.11480 \ 10^{-6}$	$1.11465 \ 10^{-6}$

for two processes: neutral-current deep-inelastic scattering and $e^+e^-\to \mu^+\mu^-$ scattering near the Z^0 -peak. The influence upon the DIS results is particularly strong in the low-x, high-y region. In this region, the small-z corrections negate a sizeable portion of the $O[\alpha^2 \ln^2(Q^2/m_e^2)]$ and higher-order contributions. The effect diminishes as x becomes larger but still remains important near $y\approx 1$. The incorporation of these corrections is therefore important in analyses of deep-inelastic data in the high y range.

The small-z resummation, on the other hand, has no visible effect upon the $e^+e^- \to \mu^+\mu^-$ cross section near the Z^0 peak. It contributes to the cross section at a level of 10^{-6} only. Correspondingly the shift in the peak cross section is negligibly small.

Acknowledgements. We would like to thank W.L. van Neerven, D.Y. Bardin, J.C. Collins, E. Laenen and D. Robaschik for useful discussions. This work was supported in part by the EC Network 'Human Capital and Mobility' under contract No. CHRX-CT923-0004 and by the German Federal Ministry for Research and Technology (BMBF) under contract No. 05 7WZ91P (0).

References

- R. Kirschner and L. Lipatov, Nucl. Phys. **B213** (1983) 122.
- V.G. Gorshkov, Uspekhi Fiz. Nauk, 110 (1973) 45, and references therein; R. Kirschner and L. Lipatov, Sov. Phys. JETP 56 (1982) 488.
- F. Berends, W. van Neerven, and G. Burgers, Nucl. Phys. B297 (1988) 429; E: B304 (1988) 921.
- J. Blümlein and A. Vogt, Acta Phys. Polonica B27 (1996) 1309.
- G. Curci, W. Furmanski, and R. Petronzio, Nucl. Phys. B175 (1980) 27.
- A.H. Mueller, Phys. Rev. **D20** (1979) 2037; H. Contopanagos, E. Laenen, and G. Sterman, ITP-SB-96-17; hep-ph/9604313.
- E.C.G. Stueckelberg and A. Peterman, Helv. Phys. Acta 26 (1953) 499;
 - M. Gell-Mann and F. Low, Phys. Rev. 95 (1954) 1300;
 L.V. Ovsyannikov, Dokl. Akad. Nauk SSSR 109 (1956) 1112;
 K. Symanzik, Comm. Math. Phys. 18 (1970) 227;
 C.G. Callan, Jr., Phys. Rev. D2 (1970) 1541;
 A. Peterman, Phys. Rep. 53 (1979) and references therein.

- 8. J.C. Collins, D.E. Soper, and G. Sterman, in: Perturbative Quantum Chromodynamics, ed. A.H. Mueller, (World Scientific, Singapore, 1989), p. 1, and references therein.
- 9. K.G. Wilson, Phys. Rev. 179 (1969) 1499.
- N.N. Bogoliubov, A.N. Tavkhelidze, and V.S. Vladimirov, Teor. Mat. Fiz. 12 (1972) 305; B.I. Zavyalov, Teor. Mat. Fiz. 17 (1973) 178, 19 (1974) 163; E. Brüning and P. Stichel, Comm. Math. Phys. 36 (1974) 137; S.A. Anikin and O.I. Zavyalov, Ann. Phys. (NY) 116 (1978) 135; B. Geyer, D. Robaschik, and E. Wieczorek, Fortschr. Physik 27 (1979) 75.
- D. Gross and F. Wilczek, Phys. Rev. **D8** (1973) 3633;
 D9 (1974) 980; H. Georgi and D. Politzer, Phys. Rev. **D9** (1974) 1829.
- 12. G. Bonneau and F. Martin, Nucl. Phys. B27 (1971) 381.
- V.S. Fadin and E.A. Kuraev, Sov. J. Nucl. Phys. 41 (1985)
 466.
- J. Blümlein and A. Vogt, Phys. Lett. **B370** (1996) 149;
 J. Blümlein and A. Vogt, Phys. Lett. **B386** (1996) 350;
 J. Blümlein, S. Riemersma, and A. Vogt, Nucl. Phys. **B** (Proc. Suppl.) **51C** (1996) 30.
- W. Beenakker, F. Berends, and W.L. van Neerven: Proceedings of the Ringberg Workshop 1989, ed. J.H. Kühn, (Springer, Berlin, 1989), p. 3; J. Blümlein, Z. Physik C47 (1990) 89.
- J. Kripfganz, H.-J. Möhring, and H. Spiesberger, Z. Physik C49 (1991) 501; J. Blümlein, Z. Physik C65 (1995) 293.
- M. Jezabek, Z. Phys. C56 (1992) 285; M. Skrzypek, Acta Phys. Polonica B23 (1992) 135.
- 18. D. Bardin, C. Burdik, P. Christova, and T. Riemann, Z. Phys. C42 (1989) 679; H. Spiesberger, in: Proceedings of the XIII Int. School of Theoretical Physics, Szczyrk, Poland, 1989, eds. M. Zralek and R. Mańka, (Nova Science, Commack, 1992), p. 145; A. Arbuzov, D. Bardin, J. Blümlein, L. Kalinovskaya, and T. Riemann, Comp. Phys. Commun. 94 (1996) 128.
- D. Bardin et al., code ZFITTER 4.8, CERN-TH-6443-92, hep-ph/9412201; and in preparation.